MATH 245 F18, Exam 2 Solutions

1. Carefully define the following terms: Proof by Contradiction theorem, Proof by Cases theorem, Proof by Induction, Proof by Reindexed Induction.

   Let p, q be propositions. The Proof by Contradiction theorem tells us that if $p \wedge \neg q \equiv F$, then $p \to q$ is true. Let p, q be propositions. The Proof by Cases theorem tells us that if there are propositions $c_1, c_2, \ldots, c_k$ with $c_1 \vee c_2 \vee \cdots \vee c_k \equiv T$, and each of $(p \wedge c_1) \to q$, $(p \wedge c_2) \to q$, $\ldots$, $(p \wedge c_k) \to q$, then $p \to q$ is true. To prove $\forall x \in \mathbb{N}\ P(x)$ by induction, we must (a) Prove $P(1)$; and (b) Prove $\forall x \in \mathbb{N}$, $P(x) \to P(x+1)$. To prove $\forall x \in \mathbb{N}\ P(x)$ by reindexed induction, we must (a) Prove $P(1)$; and (b) Prove $\forall x \in \mathbb{N}$ with $x \geq 2$, $P(x-1) \to P(x)$.

2. Carefully define the following terms: well-ordered, recurrence, big Omega, big Theta.

   Let S be a set of numbers, with an ordering $<$. We say that S is well-ordered by $<$ if every nonempty subset of S has a minimum element according to $<$. A sequence is a recurrence if all but finitely many of its terms are defined in terms of its previous terms. Given two sequences a_n and b_n, we say that a_n is big Omega of b_n to mean $\exists n_0 \in \mathbb{N},\ \exists M \in \mathbb{R},\ \forall n \geq n_0,\ M|a_n| \geq |b_n|$. Given two sequences a_n and b_n, we say that a_n is big Theta of b_n to mean that a_n is big O of b_n and also a_n is big Omega of b_n.

3. Suppose that an algorithm has runtime specified by the recurrence relation $T_n = 2nT_{n/2} + 3$. Determine what, if anything, the Master Theorem tells us.

   Because $2n$ is not a constant, the Master theorem does not apply.

4. Use induction to prove that, for all $n \in \mathbb{N}$, $\frac{(2n)!}{n!n!} \geq 2^n$.

   Base case: $n = 1$. $\frac{(2\cdot 1)!}{1!1!} = 2$, while $2^1 = 2$. Verified.

   Inductive case: Let $n \in \mathbb{N}$, and assume that $\frac{(2n)!}{n!n!} \geq 2^n$. Multiply by $\frac{(2n+2)(2n+1)}{(n+1)(n+1)}$. We get $\frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \frac{(2n)!}{n!n!} \geq \frac{(2n+2)(2n+1)}{(n+1)(n+1)} 2^n = \frac{2(2n+1)}{n+1} 2^n = \frac{2n+1}{n+1} 2^{n+1} = \frac{(n+1)+n}{n+1} 2^{n+1} = (1 + \frac{n}{n+1}) 2^{n+1} \geq 2^{n+1}$. Thus $\frac{(2(n+1))!}{(n+1)!(n+1)!} \geq 2^{n+1}$.

5. Let $a_n = n^{1.9} + n^2$. Prove that $a_n = O(n^2)$.

   Take $n_0 = 1$ and $M = 2$. For all $n \geq n_0$, we have $n^{0.1} \geq 1 = n^0$, so $n^2 \geq n^{1.9}$. Hence $a_n \leq n^2 + n^2$, and thus $|a_n| = a_n \leq 2n^2 = 2|n^2|$.

6. Let $x \in \mathbb{R}$. Prove that there is at most one $n \in \mathbb{Z}$ with $n - \frac{1}{2} \leq x < n + \frac{1}{2}$. Do not use any theorems about floors or ceilings.

   Suppose that there are $m, n \in \mathbb{Z}$ with $n - \frac{1}{2} \leq x < n + \frac{1}{2}$ and $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Hence $n - \frac{1}{2} \leq x < m + \frac{1}{2}$. Adding $\frac{1}{2}$ to both sides, we get $n < m + 1$. But also $m - \frac{1}{2} \leq x < n + \frac{1}{2}$. Subtracting $\frac{1}{2}$ from both sides, we get $m - 1 < n$. Hence $m - 1 < n < m + 1$. By Thm 1.12 in the book, since $m, n \in \mathbb{Z}$, in fact $m = n$.

7. Let $x \in \mathbb{R}$. Prove that there is at least one $n \in \mathbb{Z}$ with $n - \frac{1}{2} \leq x < n + \frac{1}{2}$. Do not use any theorems about floors or ceilings.

We use maximum element induction. Define $S = \{m \in \mathbb{Z} : m - \frac{1}{2} \leq x\}$, a nonempty set of integers with $x + \frac{1}{2}$ as an upper bound. Hence S has some maximum element n. $n - \frac{1}{2} \leq x$ because $n \in S$. We have two cases: if $x < n + \frac{1}{2}$, we are done. If instead $x \geq n + \frac{1}{2}$, then $n + 1$ is an integer, and satisfies $(n+1) - \frac{1}{2} \leq x$, so $n + 1 \in S$. But then n was the maximum element of S, a contradiction. Hence $n - \frac{1}{2} \leq x < n + \frac{1}{2}$.

8. Solve the recurrence, with initial conditions $a_0 = 3, a_1 = 4$, and relation $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 2)$.

This has characteristic polynomial $r^2 = 4r - 4$, which factors as $(r-2)^2 = 0$. Hence we have a double root, and the general solution is $a_n = A2^n + Bn2^n$. Applying our initial conditions gives $3 = a_0 = A2^0 + B \cdot 0 \cdot 2^0 = A$, and $4 = a_1 = A2^1 + B \cdot 1 \cdot 2^1 = 2A + 2B$. The system of equations $\{3 = A, 4 = 2A + 2B\}$ has solution $\{A = 3, B = -1\}$, so the specific solution is $a_n = 3 \cdot 2^n - n \cdot 2^n = (3-n)2^n$.

9. The Tribonacci numbers are given by initial conditions $T_0 = 0, T_1 = 1, T_2 = 1$, and recurrence relation $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ $(k \geq 3)$. Prove that, for all $k \in \mathbb{N}$, $T_k < 2^k$.

We handle the three base cases $k = 0, 1, 2$ separately: $T_0 = 0 < 1 = 2^0$, $T_1 = 1 < 2 = 2^1$, $T_2 = 1 < 4 = 2^2$. We now use strong induction. Let $k \in \mathbb{N}$ with $k \geq 3$. Assume that $T_{k-1} < 2^{k-1}, T_{k-2} < 2^{k-2}, T_{k-3} < 2^{k-3}$. Now, since $k \geq 3$, $T_k = T_{k-1} + T_{k-2} + T_{k-3} < 2^{k-1} + 2^{k-2} + 2^{k-3} < 2^{k-1} + 2^{k-2} + 2^{k-3} + \underbrace{2^{k-3}} = 2^{k-1} + 2^{k-2} + 2^{k-2} = 2^{k-1} + 2^{k-1} = 2^k$. Hence $T_k < 2^k$.

10. Prove that $\sqrt{3}$ is irrational.

We argue by contradiction. Suppose that $\sqrt{3}$ is rational. Hence we may assume there are $m, n \in \mathbb{Z}$, with $n \neq 0$, and $\sqrt{3} = \frac{m}{n}$. By cancelling any common factors, we may also assume that m, n have no common factors. Squaring, we get $3 = \frac{m^2}{n^2}$ and hence $3n^2 = m^2$. Now, $3|m^2$, and 3 is prime, so $3|m$ (or $3|m$). Write $m = 3k$, for some integer k, and substitute back. We get $3n^2 = (3k)^2 = 9k^2$. Hence $n^2 = 3k^2$. Again, $3|n^2$, and 3 is prime, so $3|n$ (or $3|n$). Hence m, n both have the common factor 3, a contradiction.